On Some Properties and Sums of Pell, Pell-Lucas and Modified Pell Sequences

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Abstract

In this paper, we study and show some properties and relationship of Pell, Pell-Lucas and Modified Pell sequences. Moreover, some summation formulas of $\sum_{i=1}^{n} P_{2i+2}q_{2i}$, $\sum_{i=1}^{n} P_{2i}q_{2i+2}$, $\sum_{k=0}^{n} Q_{2k}$, $\sum_{k=0}^{n} P_k^2$, $\sum_{k=0}^{n} Q_k^2$ and $\sum_{k=0}^{n} Q_{2k+1}$ are given by these properties.

Keywords: *sequences, summation formulas, Pell numbers, Pell-Lucas numbers, Modified Pell numbers*

1. Introduction

The field of sequence has been of substantial studied for many researchers [1-7]. The Pell, Pell-Lucas and Modifield Pell sequences are an interesting sequence for many mathematician [8 -1 2]. Halici and Dasdemir (2010) studied and found some relationship among Pell, Pell-Lucas and Modifield Pell sequences. After that, Halici (2011) considered some formulas for products of these sequences. Srisawat and Sripad (2016) obtained some identities for Pell and Pell-Lucas by using matrix methods. Moreover, they presented the solution of some Diophantine equation by applying these identities.

This article is organized in the following manner. Some basic known Definition and results are shown in section 2. The goal of this paper, presented in section 3, is to establish some relationship of Pell, Pell-Lucas and Modified Pell numbers. Moreover we used the first results to find some summation of these sequences.

2. Preliminaries

Throughout this paper, let P_n, Q_n and q_n are n^{th} Pell, Pell-Lucas and Modified Pell numbers respectively. In this section, the basic known Definitions and results that will be used in our work will be given. We begin this section by the following Definition.

Definition 2.1 [6] Pell sequence is a sequence defined by $P_n = 2P_{n-1} + P_{n-2}$ for any positive integer $n \geq 2$, where $P_0 = 0, P_1 = 1$.

Definition 2.2 [6] Pell-Lucas sequence is a sequence defined by $Q_n = 2Q_{n-1} + Q_{n-2}$ for any positive integer $n \geq 2$, where $Q_0 = 2, Q_1 = 2$.

Definition 2.3 [7] Modified Pell sequence is a sequence defined by $q_n = 2q_{n-1} + q_{n-2}$ for any positive integer $n \geq 2$, where $q_0 = 1, q_1 = 1$.

Proposition 2.4 [7] Let P_n and q_n are n^{th} Pell and Modified Pell numbers respectively. Thus for any positive integer n, $q_n = P_n + P_{n-1} = P_{n+1} - P_n = \frac{P_{n+1}+P_{n-1}}{2}$, $Q_n = 2(P_{n+1} - P_n) = 2(P_{n+1} - P_n) = P_{n+1} + P_{n-1}$, and $P_{n+1} = \frac{q_{n+1}+q_n}{2}$.

Lemma 2.5 [8] Let P_n and Q_n are n^{th} Pell and Pell-Lucas numbers respectively. Thus for any positive integer n, $Q_n^2 = 8P_n^2 + 4(-1)^n$.

Proposition 2.6 [10] Let P_n, Q_n and q_n are n^{th} Pell, Pell-Lucas and Modified Pell numbers respectively. Thus for any positive integer n,

1. $P_n q_{n+2} = \frac{1}{2} P_{2n+2} - (-1)^n$
2. $\sum_{i=1}^{n} P_{2i} q_{2i+2} = \frac{1}{4} P_{2n} P_{2n+4} - n$
3. $\sum_{i=1}^{n} P_{2i} Q_{2i+2} = \frac{1}{2} P_{2n} P_{2n+4} - 2n$
4. $\sum_{i=1}^{n} P_{2i}^2 = \frac{1}{8}(P_{2n} q_{2n+2} - 2n) = \frac{1}{16}(P_{2n} Q_{2n+2} - 4n)$

Theorem 2.7 [11] Let P_n, Q_n and q_n are n^{th} Pell, Pell-Lucas and Modified Pell numbers respectively. Thus for any positive integer n,

1. $P_n = \frac{Q_{n+1}+Q_{n-1}}{8}, P_n^2 = \frac{Q_{2n}+2(-1)^{n+1}}{8}$
2. $Q_n^2 = 2(q_{2n} + (-1)^n) = Q_{2n} + 2(-1)^n$, $q_n^2 = \frac{q_{2n}+(-1)^n}{2}$
3. $P_n Q_{n+1} - P_{n+1} Q_n = 2(-1)^{n+1}$
4. $P_n q_{n+1} + P_{n+1} q_n = P_{2n+1}$

The Binet's formula for P_n, Q_n and q_n are $P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, Q_n = \alpha^n + \beta^n$ and $q_n = \frac{\alpha^n + \beta^n}{\alpha + \beta}$, where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ are the roots of characteristic equation $x^2 - 2x - 1 = 0$.

3. Results

Lemma 3.1 Let P_n, Q_n and q_n are n^{th} Pell, Pell-Lucas and Modified Pell numbers respectively. Thus for any positive integer n, we have

1. $P_{n+k} q_n = \frac{1}{2}[P_{2n+k} + (-1)^n P_k]$
2. $P_n q_{n+k} = \frac{1}{2}[P_{2n+k} - (-1)^n P_k]$
3. $P_{n+k} Q_n = P_{2n+k} + (-1)^n P_k$
4. $P_n Q_{n+k} = P_{2n+k} - (-1)^n P_k$.

Proof. By using the Binet formulas of Pell, Pell-Lucas and Modified Pell numbers, we get,

1. $P_{n+k} q_n = \left(\frac{\alpha^{n+k} - \beta^{n+k}}{\alpha - \beta}\right)\left(\frac{\alpha^n + \beta^n}{\alpha + \beta}\right)$

$$= \frac{\alpha^{2n+k} + \alpha^{n+k}\beta^n - \alpha^n \beta^{n+k} - \beta^{2n+k}}{(\alpha - \beta)(\alpha + \beta)}$$

$$= \frac{\alpha^{2n+k} - \beta^{2n+k}}{2(\alpha - \beta)} + \frac{(\alpha\beta)^n(\alpha^k - \beta^k)}{2(\alpha - \beta)}$$

$$= \frac{1}{2}\left[\frac{\alpha^{2n+k}-\beta^{2n+k}}{(\alpha-\beta)} + \frac{(-1)^n(\alpha^k-\beta^k)}{(\alpha-\beta)}\right]$$
$$= \frac{1}{2}[P_{2n+k} + (-1)^n P_k],$$

and

2. $$P_n q_{n+k} = \left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)\left(\frac{\alpha^{n+k}+\beta^{n+k}}{\alpha+\beta}\right)$$
$$= \frac{\alpha^{2n+k}+\alpha^n\beta^{n+k}-\alpha^{n+k}\beta^n-\beta^{2n+k}}{(\alpha-\beta)(\alpha+\beta)}$$
$$= \frac{\alpha^{2n+k}-\beta^{2n+k}}{2(\alpha-\beta)} - \frac{(\alpha\beta)^n(\alpha^k-\beta^k)}{2(\alpha-\beta)}$$
$$= \frac{1}{2}\left[\frac{\alpha^{2n+k}-\beta^{2n+k}}{\alpha-\beta} - (-1)^n\frac{(\alpha^k-\beta^k)}{\alpha-\beta}\right]$$
$$= \frac{1}{2}[P_{2n+k} - (-1)^n P_k].$$

Since $Q_n = 2q_n$, we also have $P_{n+k}Q_n = P_{2n+k} + (-1)^n P_k$ and $P_n Q_{n+k} = P_{2n+k} - (-1)^n P_k$. The proof is completed.

Theorem 3.2 Let P_n, Q_n and q_n are n^{th} Pell, Pell-Lucas and Modified Pell numbers respectively. Thus for any positive integer n, we have

1. $\sum_{i=1}^{n} P_{2i+2}q_{2i} = \frac{1}{4}P_{2n+4}P_{2n} + n$
2. $\sum_{i=1}^{n} P_{2i+2}Q_{2i} = \frac{1}{2}P_{2n+4}P_{2n} + 2n$
3. $\sum_{i=1}^{n} P_{2i}q_{2i+2} = \frac{1}{4}P_{2n+4}P_{2n} - n$
4. $\sum_{i=1}^{n} P_{2i}Q_{2i+2} = \frac{1}{2}P_{2n+4}P_{2n} - 2n$

Proof. Firstly, let us define a sequence as follows;

$$a_n = \left[\frac{1}{4}P_{2n+4}P_{2n} + n\right] - \left[\frac{1}{4}P_{2n+2}P_{2n-2} + (n-1)\right]$$
$$= \frac{1}{4}[P_{2n+4}P_{2n} - P_{2n+2}P_{2n-2} + 4]$$
$$= \frac{1}{4}\left[\left(\frac{\alpha^{2n+4}-\beta^{2n+4}}{\alpha-\beta}\right)\left(\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}\right) - \left(\frac{\alpha^{2n+2}-\beta^{2n+2}}{\alpha-\beta}\right)\left(\frac{\alpha^{2n-2}-\beta^{2n-2}}{\alpha-\beta}\right) + 4\right]$$
$$= \frac{1}{4}\left[\frac{\alpha^{4n}(\alpha^4-1)+\beta^{4n}(\beta^4-1)-(\alpha\beta)^{2n}(\alpha^4+\beta^4)+(\alpha\beta)^{2n-2}(\alpha^4+\beta^4)}{(2\sqrt{2})(\alpha-\beta)} + 4\right]$$
$$= \frac{1}{4}\left[\frac{4\sqrt{2}(\alpha^{4n+2}-\beta^{4n+2})}{(2\sqrt{2})(\alpha-\beta)} + 4\right] = \frac{1}{2}\left[\frac{\alpha^{4n+2}-\beta^{4n+2}}{\alpha-\beta} + 2\right]$$
$$= \frac{1}{2}[P_{4n+2} + 2] = \frac{1}{2}\left[P_{2(2n)+2} + (-1)^{2n}P_2\right]$$
$$= P_{2n+2}q_{2n} \quad \text{(By Lemma 3.1(1))}$$

Thus, we have;

1. $$\sum_{i=1}^{n} P_{2i+2}q_{2i} = \sum_{i=1}^{n}\left[\left(\frac{1}{4}P_{2i+4}P_{2i} + i\right) - \left(\frac{1}{4}P_{2i+2}P_{2i-2} + (i-1)\right)\right]$$
$$= \left[\left(\frac{1}{4}P_6P_2 + 1\right) - \left(\frac{1}{4}P_4P_0 + 0\right)\right] + \left[\left(\frac{1}{4}P_8P_4 + 2\right) - \left(\frac{1}{4}P_6P_2 + 1\right)\right]$$
$$+ \left[\left(\frac{1}{4}P_{10}P_6 + 3\right) - \left(\frac{1}{4}P_8P_4 + 2\right)\right] + \left[\left(\frac{1}{4}P_{12}P_8 + 4\right) - \left(\frac{1}{4}P_{10}P_6 + 3\right)\right]$$

$$+\cdots+\left[\left(\frac{1}{4}P_{2(n-2)+4}P_{2(n-2)}+(n-2)\right)-\left(\frac{1}{4}P_{2(n-2)+2}P_{2(n-2)-2}+(n-3)\right)\right]$$
$$+\left[\left(\frac{1}{4}P_{2(n-1)+4}P_{2(n-1)}+(n-1)\right)-\left(\frac{1}{4}P_{2(n-1)+2}P_{2(n-1)-2}+(n-2)\right)\right]$$
$$+\left[\left(\frac{1}{4}P_{2n+4}P_{2n}+n\right)-\left(\frac{1}{4}P_{2n+2}P_{2n-2}+(n-1)\right)\right]$$
$$=\left(\frac{1}{4}P_{2n+4}P_{2n}+n\right)-\left(\frac{1}{4}P_4P_0+0\right)$$
$$=\frac{1}{4}P_{2n+4}P_{2n}+n.$$

2. Since $Q_n=2q_n$, it follows that $\sum_{i=1}^{n}P_{2i+2}Q_{2i}=\frac{1}{2}P_{2n+4}P_{2n}+2n$.

3. Let $b_n=\left[\frac{1}{4}P_{2n+4}P_{2n}+n\right]-\left[\frac{1}{4}P_{2n+2}P_{2n-2}+(n+1)\right]$, thus we have $b_n=\frac{1}{4}[P_{2n+4}P_{2n}-P_{2n+2}P_{2n-2}-4]$. Similarly to the prove of a_n and by Lemma 3.1(2), we have $b_n=P_{2n}q_{2n+2}$. It follows that,

$$\sum_{i=1}^{n}P_{2i}q_{2i+2}=\sum_{i=1}^{n}\left[\left(\frac{1}{4}P_{2i+4}P_{2i}+i\right)-\left(\frac{1}{4}P_{2i+2}P_{2i-2}+(i+1)\right)\right]$$
$$=\left[\left(\frac{1}{4}P_6P_2+1\right)-\left(\frac{1}{4}P_4P_0+2\right)\right]+\left[\left(\frac{1}{4}P_8P_4+2\right)-\left(\frac{1}{4}P_6P_2+3\right)\right]$$
$$+\left[\left(\frac{1}{4}P_{10}P_6+3\right)-\left(\frac{1}{4}P_8P_4+4\right)\right]+\left[\left(\frac{1}{4}P_{12}P_8+4\right)-\left(\frac{1}{4}P_{10}P_6+5\right)\right]+\cdots$$
$$+\left[\left(\frac{1}{4}P_{2(n-2)+4}P_{2(n-2)}+(n-2)\right)-\left(\frac{1}{4}P_{2(n-2)+2}P_{2(n-2)-2}+(n-2)+1\right)\right]$$
$$+\left[\left(\frac{1}{4}P_{2(n-1)+4}P_{2(n-1)}+(n-1)\right)-\left(\frac{1}{4}P_{2(n-1)+2}P_{2(n-1)-2}+(n-1)+1\right)\right]$$
$$+\left[\left(\frac{1}{4}P_{2n+4}P_{2n}+n\right)-\left(\frac{1}{4}P_{2n+2}P_{2n-2}+(n+1)\right)\right]$$
$$=\frac{1}{4}P_{2n+4}P_{2n}-\frac{1}{4}P_4P_0-n$$
$$=\frac{1}{4}P_{2n+4}P_{2n}-n$$

4. Since $Q_n=2q_n$, it follows that $\sum_{i=1}^{n}P_{2i}Q_{2i+2}=\frac{1}{2}P_{2n+4}P_{2n}-2n$.

Lemma 3.3 Let P_n, Q_n and q_n are n^{th} Pell, Pell-Lucas and Modified Pell numbers respectively. Thus for any positive integer n, we have

1. $Q_{2n}=P_{n+2}q_n-P_nq_{n-2}, n\geq 2$
2. $Q_{2n}=P_{n-1}q_{n+3}-P_{n-3}q_{n+1}, n\geq 3$.

Proof. 1. Considering Lemma 3.1(1), we have

$$P_{n+2}q_n-P_nq_{n-2}=\frac{1}{2}[P_{2n+2}+(-1)^nP_2]-\frac{1}{2}[P_{2(n-2)+2}+(-1)^{n-2}P_2]$$
$$=\frac{1}{2}[P_{2n+2}+(-1)^nP_2]-\frac{1}{2}[P_{2n-2}+(-1)^{n-2}P_2]$$
$$=\frac{1}{2}[P_{2n+2}-P_{2n-2}]$$
$$=\frac{1}{2}[(2P_{2n+1}+P_{2n})-P_{2n-2}]$$
$$=\frac{1}{2}[2P_{2n+1}+(2P_{2n-1}+P_{2n-2})-P_{2n-2}]$$
$$=P_{2n+1}+P_{2n-1}$$
$$=Q_{2n}$$

2. By Lemma 3.1(2), we have

$$\begin{aligned} P_{n-1}q_{n+3} - P_{n-3}q_{n+1} &= P_{(n-1)}q_{(n-1)+4} - P_{(n-3)}q_{(n-3)+4} \\ &= \frac{1}{2}\left[P_{2(n-1)+4} - (-1)^{n-1}P_4\right] - \frac{1}{2}\left[P_{2(n-3)+4} - (-1)^{n-3}P_4\right] \\ &= \frac{1}{2}\left[P_{2n+2} - P_{2n-2}\right] \\ &= Q_{2n} \end{aligned}$$

So, the proof is completed.

Theorem 3.4 Let P_n, Q_n and q_n are n^{th} Pell, Pell-Lucas and Modified Pell numbers respectively. Thus for any positive integer n, we have

1. $\sum_{k=0}^{n} Q_{2k} = P_{n+2}q_n + P_{n+1}q_{n-1} + 1$
2. $\sum_{k=0}^{n} Q_{2k} = P_{n-2}q_{n+2} + P_{n-1}q_{n+3} + 1, n \geq 2$
3. $\sum_{k=0}^{n} q_{2k} = \frac{P_{n+2}q_n + P_{n+1}q_{n-1} + 1}{2}$
4. $\sum_{k=0}^{n} q_{2k} = \frac{P_{n-2}q_{n+2} + P_{n-1}q_{n+3} + 1}{2}, n \geq 2.$

Proof. By Definition of Pell-Lucas and Lemma 3.3, we get

1.
$$\begin{aligned} \sum_{k=0}^{n} Q_{2k} &= Q_0 + Q_2 + \sum_{k=2}^{n} Q_{2k} \\ &= Q_0 + Q_2 + \sum_{k=2}^{n}(P_{k+2}q_k - P_k q_{k-2}) \\ &= Q_0 + Q_2 + (P_4q_2 - P_2q_0) + (P_5q_3 - P_3q_1) + (P_6q_4 - P_4q_2) \\ &\quad + (P_7q_5 - P_5q_3) + (P_8q_6 - P_6q_4) + \cdots + (P_{n-1}q_{n-3} - P_{n-3}q_{n-5}) \\ &\quad + (P_nq_{n-2} - P_{n-2}q_{n-4}) + (P_{n+1}q_{n-1} - P_{n-1}q_{n-3}) + (P_{n+2}q_n - P_nq_{n-2}) \\ &= Q_0 + Q_2 + P_{n+2}q_n + P_{n+1}q_{n-1} - P_3q_1 - P_2q_0 \\ &= 2 + 6 + P_{n+2}q_n + P_{n+1}q_{n-1} - (5)(1) - (2)(1) \\ &= P_{n+2}q_n + P_{n+1}q_{n-1} + 1. \end{aligned}$$

2.
$$\begin{aligned} \sum_{k=0}^{n} Q_{2k} &= Q_0 + Q_2 + Q_4 + \sum_{k=3}^{n} Q_{2k} \\ &= Q_0 + Q_2 + Q_4 + \sum_{k=3}^{n}(P_{k-1}q_{k+3} - P_{k-3}q_{k+1}) \\ &= Q_0 + Q_2 + Q_4 + (P_2q_6 - P_0q_4) + (P_3q_7 - P_1q_5) + (P_4q_8 - P_2q_6) \\ &\quad + (P_5q_9 - P_3q_7) + (P_6q_{10} - P_4q_8) + \cdots + (P_{n-3}q_{n+1} - P_{n-5}q_{n-1}) \\ &\quad + (P_{n-2}q_{n+2} - P_{n-4}q_n) + (P_{n-1}q_{n+3} - P_{n-3}q_{n+1}) \\ &= Q_0 + Q_2 + Q_4 + P_{n-2}q_{n+2} + P_{n-1}q_{n+3} - P_1q_5 - P_0q_4 \\ &= 42 + P_{n-2}q_{n+2} + P_{n-1}q_{n+3} - 41 \\ &= P_{n-2}q_{n+2} + P_{n-1}q_{n+3} + 1. \end{aligned}$$

For 3 and 4, since $Q_n = 2q_n$, we obtain that $\sum_{k=0}^{n} q_{2k} = \frac{P_{n+2}q_n + P_{n+1}q_{n-1} + 1}{2}$ and $\sum_{k=0}^{n} q_{2k} = \frac{P_{n-2}q_{n+2} + P_{n-1}q_{n+3} + 1}{2}, n \geq 2.$

Thus, the proof is completed.

Theorem 3.5 Let P_n and q_n are n^{th} Pell and Modified Pell numbers respectively, then for any positive integer n, we have

1. $\sum_{k=0}^{n} P_k^2 = \frac{1}{8}[P_{n+2}q_n + P_{n+1}q_{n-1} + (-1)^{n+1}]$
2. $\sum_{k=0}^{n} P_k^2 = \frac{1}{8}[P_{n-2}q_{n+2} + P_{n-1}q_{n+3} + (-1)^{n+1}], n \geq 2.$

Proof. By Theorem 2.7 and Theorem 3.4, we have

1. $\sum_{k=0}^{n} P_k^2 = \sum_{k=0}^{n} \frac{1}{8}(Q_{2k} + 2(-1)^{k+1}) = \frac{1}{8}[\sum_{k=0}^{n} Q_{2k} + 2\sum_{k=0}^{n}(-1)^{k+1}]$
$= \frac{1}{8}[P_{n+2}q_n + P_{n+1}q_{n-1} + 1 + 2\sum_{k=0}^{n}(-1)^{k+1}]$

Case 1.1 : If n is even, then we get

$$\sum_{k=0}^{n} P_k^2 = \frac{1}{8}[P_{n+2}q_n + P_{n+1}q_{n-1} + 1 + 2(-1)] = \frac{1}{8}[P_{n+2}q_n + P_{n+1}q_{n-1} - 1]$$

Case 1.2 : If n is odd, then we get

$$\sum_{k=0}^{n} P_k^2 = \frac{1}{8}[P_{n+2}q_n + P_{n+1}q_{n-1} + 1 + 2(0)] = \frac{1}{8}[P_{n+2}q_n + P_{n+1}q_{n-1} + 1]$$

Therefore, $\sum_{k=0}^{n} P_k^2 = \frac{1}{8}[P_{n+2}q_n + P_{n+1}q_{n-1} + (-1)^{n+1}]$.

2. $\sum_{k=0}^{n} P_k^2 = \sum_{k=0}^{n} \frac{1}{8}(Q_{2k} + 2(-1)^{k+1}) = \frac{1}{8}[\sum_{k=0}^{n} Q_{2k} + 2\sum_{k=0}^{n}(-1)^{k+1}]$
$= \frac{1}{8}[P_{n-2}q_{n+2} + P_{n-1}q_{n+3} + 1 + 2\sum_{k=0}^{n}(-1)^{k+1}]$

Case 2.1 : If n is even, then we get

$$\sum_{k=0}^{n} P_k^2 = \frac{1}{8}[P_{n-2}q_{n+2} + P_{n-1}q_{n+3} + 1 + 2(-1)] = \frac{1}{8}[P_{n-2}q_{n+2} + P_{n-1}q_{n+3} - 1]$$

Case 2.2 : If n is odd, then we get

$$\sum_{k=0}^{n} P_k^2 = \frac{1}{8}[P_{n-2}q_{n+2} + P_{n-1}q_{n+3} + 1 + 2(0)] = \frac{1}{8}[P_{n-2}q_{n+2} + P_{n-1}q_{n+3} + 1]$$

Therefore, $\sum_{k=0}^{n} P_k^2 = \frac{1}{8}[P_{n-2}q_{n+2} + P_{n-1}q_{n+3} + (-1)^{n+1}]$.

Theorem 3.6 Let P_n, Q_n and q_n are n^{th} Pell, Pell-Lucas and Modified Pell numbers respectively. Thus for any positive integer n, we have

1. $\sum_{k=0}^{n} Q_k^2 = \begin{cases} P_{n+2}q_n + P_{n+1}q_{n-1} + 3; & n \text{ is even number} \\ P_{n+2}q_n + P_{n+1}q_{n-1} + 1; & n \text{ is odd number} \end{cases}$

2. $\sum_{k=0}^{n} Q_k^2 = \begin{cases} P_{n-2}q_{n+2} + P_{n-1}q_{n+3} + 3; & n \text{ is even number} \\ P_{n-2}q_{n+2} + P_{n-1}q_{n+3} + 1; & n \text{ is odd number and } n \geq 3 \end{cases}$

Proof. By Lemma 2.6 and Theorem 3.3, we have

1. $\sum_{k=0}^{n} Q_k^2 = \sum_{k=0}^{n}(8P_k^2 + 4(-1)^k) = 8\sum_{k=0}^{n} P_k^2 + 4\sum_{k=0}^{n}(-1)^k$

Case 1.1 : If n is even, then we get

$$\sum_{k=0}^{n} Q_k^2 = 8\sum_{k=0}^{n} P_k^2 + 4\sum_{k=0}^{n}(-1)^k = 8\left[\frac{1}{8}(P_{n+2}q_n + P_{n+1}q_{n-1} - 1)\right] + 4 = P_{n+2}q_n + P_{n+1}q_{n-1} + 3$$

Case 1.2 : If n is odd, then we get

$$\sum_{k=0}^{n} Q_k^2 = 8\sum_{k=0}^{n} P_k^2 + 4\sum_{k=0}^{n}(-1)^k$$

$$= 8\left[\frac{1}{8}(P_{n+2}q_n + P_{n+1}q_{n-1} + 1)\right] + 4(0)$$
$$= P_{n+2}q_n + P_{n+1}q_{n-1} + 1$$

2. $\sum_{k=0}^{n} Q_k^2 = \sum_{k=0}^{n}(8P_k^2 + 4(-1)^k) = 8\sum_{k=0}^{n} P_k^2 + 4\sum_{k=0}^{n}(-1)^k$

Case 2.1 : If n is even, then we get

$$\sum_{k=0}^{n} Q_k^2 = 8\sum_{k=0}^{n} P_k^2 + 4\sum_{k=0}^{n}(-1)^k$$
$$= 8\left[\frac{1}{8}(P_{n-2}q_{n+2} + P_{n-1}q_{n+3} - 1)\right] + 4$$
$$= P_{n-2}q_{n+2} + P_{n-1}q_{n+3} + 3$$

Case 2.2 : If n is odd, then we get

$$\sum_{k=0}^{n} Q_k^2 = 8\sum_{k=0}^{n} P_k^2 + 4\sum_{k=0}^{n}(-1)^k$$
$$= 8\left[\frac{1}{8}(P_{n-2}q_{n+2} + P_{n-1}q_{n+3} + 1)\right] + 4(0)$$
$$= P_{n-2}q_{n+2} + P_{n-1}q_{n+3} + 1$$

Lemma 3.7 Let P_n, Q_n and q_n are n^{th} Pell, Pell-Lucas and Modified Pell numbers respectively. Thus for any positive integer n, we have

1. $Q_{2n+1} = P_{n+2}q_{n+1} - P_nq_{n-1}$
2. $Q_{2n+1} = P_{n+1}q_{n+2} - P_{n-1}q_n$

Proof. 1. By Proposition 2.5 and Lemma 3.1(1), we have

$$P_{n+2}q_{n+1} - P_nq_{n-1} = P_{(n+1)+1}q_{n+1} - P_{(n-1)+1}q_{n-1}$$
$$= \frac{1}{2}\left[P_{2(n+1)+1} + (-1)^{n+1}P_1\right] - \frac{1}{2}\left[P_{2(n-1)+1} + (-1)^{n-1}P_1\right]$$
$$= \frac{1}{2}[P_{2n+3} - P_{2n-1}]$$
$$= \frac{1}{2}[2P_{2n+2} + P_{2n+1} + 2P_{2n} - P_{2n+1}]$$
$$= P_{2n+2} + P_{2n}$$
$$= Q_{2n+1}$$

2. By Proposition 2.5 and Lemma 3.1(2), we have

$$P_{n+1}q_{n+2} - P_{n-1}q_n = P_{n+1}q_{(n+1)+1} - P_{n-1}q_{(n-1)+1}$$
$$= \frac{1}{2}\left[P_{2(n+1)+1} - (-1)^{n+1}P_1\right] - \frac{1}{2}\left[P_{2(n-1)+1} - (-1)^{n-1}P_1\right]$$
$$= \frac{1}{2}[P_{2n+3} - P_{2n-1}]$$
$$= Q_{2n+1}$$

Theorem 3.8 Let P_n, Q_n and q_n are n^{th} Pell, Pell-Lucas and Modified Pell numbers respectively. Thus for any positive integer n, we have

1. $\sum_{k=0}^{n} Q_{2k+1} = P_{n+1}q_n + P_{n+2}q_{n+1} - 1$
2. $\sum_{k=0}^{n} Q_{2k+1} = P_nq_{n+1} + P_{n+1}q_{n+2} - 1.$

Proof. By Definition of Pell-Lucas and Lemma 3.7, we get

1. $\sum_{k=0}^{n} Q_{2k+1} = Q_1 + \sum_{k=1}^{n} Q_{2k+1}$

$$= Q_1 + \sum_{k=1}^{n}(P_{k+2}q_{k+1} - P_kq_{k-1})$$
$$= Q_1 + (P_3q_2 - P_1q_0) + (P_4q_3 - P_2q_1) + (P_5q_4 - P_3q_2)$$
$$+ \cdots + (P_{n-1}q_{n-2} - P_{n-3}q_{n-4}) + (P_nq_{n-1} - P_{n-2}q_{n-3})$$
$$+ (P_{n+1}q_n - P_{n-1}q_{n-2}) + (P_{n+2}q_{n+1} - P_nq_{n-1})$$

$$= Q_1 + P_{n+1}q_n + P_{n+2}q_{n+1} - P_2q_1 - P_1q_0$$
$$= 2 + P_{n+1}q_n + P_{n+2}q_{n+1} - (2)(1) - (1)(1)$$
$$= P_{n+1}q_n + P_{n+2}q_{n+1} - 1$$

2. $\sum_{k=0}^{n} Q_{2k+1} = Q_1 + \sum_{k=1}^{n} Q_{2k+1}$

$$= Q_1 + \sum_{k=1}^{n}(P_{k+1}q_{k+2} - P_{k-1}q_k)$$
$$= Q_1 + (P_2q_3 - P_0q_1) + (P_3q_4 - P_1q_2) + (P_4q_5 - P_2q_3)$$
$$+ \cdots + (P_{n-2}q_{n-1} - P_{n-4}q_{n-3}) + (P_{n-1}q_n - P_{n-3}q_{n-2})$$
$$+ (P_nq_{n+1} - P_{n-2}q_{n-1}) + (P_{n+1}q_{n+2} - P_{n-1}q_n)$$
$$= Q_1 + P_nq_{n+1} + P_{n+1}q_{n+2} - P_1q_2 - P_0q_1$$
$$= 2 + P_nq_{n+1} + P_{n+1}q_{n+2} - (1)(3) - (0)(1)$$
$$= P_nq_{n+1} + P_{n+1}q_{n+2} - 1.$$

Thus, the proof is now completed.

4. Conclusions

In this paper we shown some relationship of Pell, Pell-Lucas and Modified Pell numbers. Moreover, we gave some formula of $\sum_{i=1}^{n} P_{2i+2}q_{2i}$, $\sum_{i=1}^{n} P_{2i}q_{2i+2}$, $\sum_{k=0}^{n} Q_{2k}$, $\sum_{k=0}^{n} P_k^2$, $\sum_{k=0}^{n} Q_k^2$ and $\sum_{k=0}^{n} Q_{2k+1}$.

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